JOURNAL OF COMBINATORIAL THEORY (B) 20, 185–203 (1976)

n-Tuple Colorings and Associated Graphs

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Communicated by W. T. Tutte

Received August 8, 1973

The n-tuple graph coloring, which assigns to each vertex n colors, is defined together with its respective chromatic number χ_n. It is proved that these numbers satisfy the inequality $\chi_n \geqslant 2 + \chi_{n-1}$, and that equality holds only for bipartite graphs. Graphs $G_n{}^m$ are defined which play the same role for the n-tuple coloring that K_m plays for the conventional coloring. The chromatic numbers of various classes of graphs are also calculated.

I. DEFINITIONS AND INTRODUCTION

A graph G is understood to be finite and to contain no loops or multiple edges. Its vertex set is denoted by $V(G)$. C_n and K_n represent, respectively, the n-cycle and the complete graph on n vertices. The cardinality of a set S is denoted by $|S|$.

I^+ is the set of all positive integers and I^m is the set of all positive integers not greater than m. $I_n{}^m$ is the family of all subsets of I^m of cardinality n. We set the convention that when any member of $I_n{}^m$ is written out explicitely, its elements should be listed in monotone increasing order. The graph $G_n{}^m$ is now defined as follows, $V(G_n{}^m) = I_n{}^m$ and two vertices are adjacent iff they are disjoint. For example, Fig. 1 is a drawing of $G_2{}^5$.

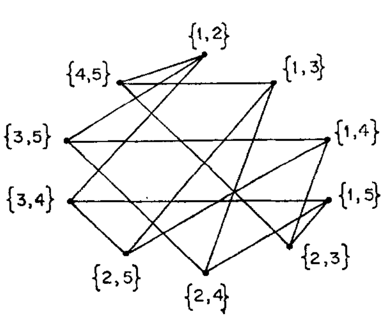


FIGURE 1

These graphs were originally defined and to some extent discussed in [2]. If $m < 2n$ then $G_n{}^m$ has no edges and is of no interest to us. Consequently, whenever a graph $G_n{}^m$ is considered in the sequel, it is implicitly assumed that $m \geqslant 2n$. If $n = 1$, then any two vertices are adjacent and $G_1{}^m = K_m$.

An *n-tuple coloring* of a graph G is an assignment of n distinct colors to each vertex of G in such a manner that no two adjacent vertices share a color. This is clearly a generalization of the conventional coloring, which assigns only one color to each vertex. The n-tuple coloring can be regarded from two different points of view, and both will be used frequently in this paper. The first of these makes use of the notion of homomorphism. We recall that a map η: $V(G) \to V(H)$ is said to be a *homomorphism of G into H if* η maps pairs of adjacent vertices of G onto pairs of adjacent vertices of H. Now suppose that an n-tuple coloring of G utilizes a total of m colors. We may then denote the colors used in this coloring by $1, 2,..., m$. The color assignment then becomes a mapping η: $V(G) \to I_n{}^m$. But $I_n{}^m$ is the vertex set of $G_n{}^m$ and the condition that no two adjacent vertices share a color makes η into a homomorphism of G into $G_n{}^m$. To see this the reader should recall that two vertices of $G_n{}^m$ are adjacent iff they are disjoint. Thus, we have obtained our first reformulation of the concept of an n-tuple coloring:

AF1. *An n-tuple coloring of G with m colors is a homomorphism* $\eta: G \to G_n{}^m$.

This alternate formulation will be referred to as AF1.

The second point of view utilizes independent sets. A subset of $V(G)$ is said to be *independent* if it contains no adjacent vertices. We again identify the colors used with the integers of I^m. Let C_i be the set of vertices to which the "color" i was assigned. The condition that no two adjacent vertices share a color now means that each C_i is an independent set. Thus,

AF2. *An n-tuple coloring of G with m colors is a collection* $\{C_i\}$ $(1 \leqslant i \leqslant m)$ *of independent subsets of* $V(G)$ *such that every vertex of G is contained in exactly n of the* C_i.

As was defined in [2], such a family of sets is said to *cover G exactly n times*. This point of view will be referred to as AF2. It should be noted that for $n = 1$ these points of view are already known and widely used in the literature (see [3, Chap. 12]).

$\chi_n(G)$, the nth chromatic number of G, is the smallest number of colors needed to give G an n-tuple coloring. In view of AF1, $\chi_n(G)$ is the smallest integer m for which there exists a homomorphism $G \to G_n{}^m$.

Alternately, AF2 tells us that $\chi_n(G)$ is the smallest number of independent subsets of $V(G)$ which cover G exactly n times.

Obviously $\chi_1(G) = \chi(G)$, the conventional chromatic number. The higher chromatic numbers can also be given an alternate interpretation in terms of $\chi(G)$. If $G[H]$ is the lexicographic product of the graphs G and H (see [2]), then an n-tuple coloring of G is equivalent to a conventional coloring of $G[K_n]$. Therefore,

$$\chi_n(G) = \chi(G[K_n]).$$

This allows us to reinterpret [2, Theorem 3 and the corollary to Theorem 5] as

THEOREM 5′. *If G is critical and not complete then*

$$\chi_n(G) \leqslant n\chi(G) - (n/2).$$

THEOREM 3′. $\chi(G[H]) = \chi_n(G)$ *where* $n = \chi(H)$.

As these theorems will not be used in this paper, no new proof is offered.

2. GENERAL PROPERTIES OF χ_n

Lemma 3 of [2] can be reinterpreted as Theorem 1 below. As this theorem will be used repeatedly in the sequel, we offer a new proof.

THEOREM 1. $\chi_{qn+r}(G) \leqslant q\chi_n(G) + \chi_r(G)$.

Proof. Let $\{C_i\}$ and $\{C_j'\}$ be an m-tuple and an n-tuple coloring of G (see AF2). $\{C_i\} \cup \{C_j'\}$ is then an $(m + n)$-tuple coloring of G. Therefore, we have

$$\chi_{m+n}(G) \leqslant \chi_m(G) + \chi_n(G),$$

and the full statement of the theorem follows easily. Q.E.D.

Theorems 6 and 4 of [2] are both sharpened by the following very useful theorem.

THEOREM 2. *If G has an edge, then* $\chi_n(G) \geqslant 2 + \chi_{n-1}(G)$ *for all* $n > 1$.

The proof of this theorem relies on the following lemma.

LEMMA. *If* $m > 2n$, *then there exists a homomorphism* $\eta\colon G_n{}^m \to G_{n-1}^{m-2}$.

Proof. A vertex $A = \{a_1, a_2, ..., a_n\}$ of G_n^m is said to be k-regular if for some k, $a_k = k$ and $a_{k+1} > k + 1$. If a vertex is not k-regular for any k, it is said to be irregular. Since we have set the convention that $a_i < a_{i+1}$ for $1 \leqslant i < n$, it follows that in a k-regular vertex $a_i = i$ for all $i \leqslant k$ and $a_i > i$ for all other i. On the other hand, if A is irregular then $a_i > i$ for all $i \leqslant n$. We now defined the map η as follows. If A is k-regular then

$$\eta(A) = \{a_2 - 1, a_3 - 1, ..., a_k - 1, a_{k+1} - 2, a_{k+2} - 2, ..., a_n - 2\};$$

if A is irregular then

$$\eta(A) = \{a_2 - 2, a_3 - 2, ..., a_n - 2\}.$$

It is easily seen that η is in fact a mapping into I_{n-1}^{m-2}. Hence, it now remains to show that η is in fact a homomorphism of G_n^m into G_{n-1}^{m-2}. We must show that if $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ are adjacent vertices of G_n^m then $\eta(A)$ and $\eta(B)$ are adjacent vertices of G_{n-1}^{m-2}. In other words, $A \cap B = \varnothing \Rightarrow \eta(A) \cap \eta(B) = \varnothing$. So now suppose A and B are disjoint. Since any regular vertex must necessarily contain the integer 1, it follows that not both A and B are regular. The proof is therefore reduced to two cases.

Case 1. Both A and B are irregular. Then $\eta(A) = \{a_1 - 2, a_3 - 2, ..., a_n - 2\}$ and $\eta(B) = \{b_2 - 2, b_3 - 2, ..., b_n - 2\}$. So, if $\eta(A) \cap \eta(B) \neq \varnothing$, then there exists some a_i and b_j such that $a_i - 2 = b_j - 2$, or $a_i = b_j$, contradicting the disjointness of A and B.

Case 2. A is k-regular and B is irregular. Then,

$$\eta(A) = \{a_2 - 1, a_3 - 1, ..., a_k - 1, a_{k+1} - 2, a_{k+2} - 2, ..., a_n - 2\}$$

and

$$\eta(B) = \{b_2 - 2, b_3 - 2, ..., b_n - 2\}.$$

If $\eta(A) \cap \eta(B) \neq \varnothing$, then

$$\text{for some } j > 1, \qquad b_j - 2 \in \eta(A). \tag{1}$$

If $b_j - 2 = a_i - 2$ for some $i > k$, then $b_j = a_i$, contradicting the disjointness of A and B. Hence, we must have that $b_j - 2 = a_i - 1$ for some $2 \leqslant i \leqslant k$.

$$\therefore \quad b_j = a_i + 1 = i + 1 \leqslant k + 1. \tag{2}$$

However, since A is k-regular it follows that $I^k \subseteq A$. Since $A \cap B = \varnothing$, this implies that

$$b_s \geqslant k + 1 \qquad \text{for all } b_s \in B. \tag{3}$$

Combining (3) with (2) we obtain: $b_j = k + 1$. b_j is therefore the smallest element of B, i.e., $j = 1$. This, however, contradicts (1).

Thus, we have derived a contradiction from the assumption that $\eta(A) \cap \eta(B) \neq \varnothing$, and, consequently, η preserves adjacency and is a homomorphism. Q.E.D.

Proof of Theorem 2. Let uv be an edge of G and let $m = \chi_n(G)$. In view of AF1, there exists a homomorphism $\gamma: G \to G_n{}^m$. Since u and v are adjacent, $\gamma(u) \cap \gamma(v) = \varnothing$, and therefore, $|\gamma(u) \cup \gamma(v)| = 2n$. Consequently, $m \geqslant 2n$, and by the lemma there exists a homomorphism $\eta: G_n{}^m \to G_{n-1}^{m-2}$. Since the composition of two homomorphisms is also a homomorphism it follows that $\eta \circ \gamma$ is a homomorphism of G into G_{n-1}^{m-2}.

$$\therefore \quad \chi_{n-1}(G) \leqslant m - 2 = \chi_n(G) - 2.$$

Q.E.D.

COROLLARY. *If G has an edge then*

$$\chi_n(G) \geqslant \chi_k(G) + 2n - 2k \qquad \textit{if} \quad n \geqslant k.$$

The following theorem is a natural generalization of a well-known theorem on the chromatic number.

THEOREM 3. *If $\gamma: G \to H$ is a homomorphism then*

$$\chi_n(G) \leqslant \chi_n(H) \qquad \textit{for all } n \geqslant 1.$$

Proof. Let $m = \chi_n(H)$; then there exists a homomorphism $\beta: H \to G_n{}^m$.

$$\therefore \quad \beta \circ \gamma \text{ is a homomorphism of } G \text{ into } G_n{}^m.$$
$$\therefore \quad \chi_n(G) \leqslant m = \chi_n(H).$$

Q.E.D.

COROLLARY. *If the graphs G and H have homomorphisms $\alpha: G \to H$ and $\beta: H \to G$, then $\chi_n(G) = \chi_n(H)$ for all $n \geqslant 1$.*

The converse to the above corollary does not hold. We shall later see that $\chi_n(G_2{}^5) = 2n + 1 + [(n-1)/2]$. Moreover, the example near the end of Section 3 introduces another graph G with the same chromatic numbers. However, the reader can easily verify that neither G nor $G_2{}^5$

contain triangles and yet any homomorphic image of either must contain a triangle. It follows that neither G nor $G_2{}^5$ contains a homomorphic image of the other.

3. Complete n-Partite and Bipartite Graphs, C_n and G_n^{2n+1}

We are now in position to calculate the n-tuple chromatic numbers of the graphs listed in this chapter's title. A graph is said to be *complete n-partite* if it has a family $\{U_i\}$ $(1 \leqslant i \leqslant n)$ of pairwise disjoint nonvoid sets of vertices such that $\bigcup_{i=1}^{n} U_i = V(G)$ and two vertices are adjacent iff they belong to different U_i. Clearly these U_i are the maximal independent subsets of G.

THEOREM 4. *If G is complete n-partite then $\chi_k(G) = kn$.*

Proof. Clearly $\chi_1(G) = n$, and applying Theorem 1 we get

$$\chi_k(G) \leqslant kn. \tag{4}$$

On the other hand, let $\sigma = \{S_\lambda\}_{1 \leqslant \lambda \leqslant n}$ be any k-tuple coloring of G. For each $1 \leqslant i \leqslant n$, let $u_i \in U_i$ and let σ_i be the subfamily of σ which consists of all S_λ containing u_i. By AF2, $|\sigma_i| = k$. Moreover, if $S_\lambda \in \sigma_i$, then $S_\lambda \subseteq U_i$, and since the U_i are pairwise disjoint it follows that also $\sigma_i \cap \sigma_j = \varnothing$ for $i \neq j$.

$$\therefore \quad m = |\sigma| \geqslant \sum_{i=1}^{n} |\sigma_i| = kn.$$

$$\therefore \quad \chi_k(G) \geqslant kn.$$

Combining this with (4) we get $\chi_k(G) = kn$. Q.E.D.

A graph G is said to be bipartite if $\chi(G) = 2$. We shall later see that bipartite graphs are the only graphs to satisfy the extreme case in Theorem 2. For the moment, however, we must content ourselves with the calculation of their chromatic numbers.

THEOREM 5. *If G is bipartite then $\chi_n(G) = 2n$.*

Proof. From Theorem 1 it follows that $\chi_n(G) \leqslant n\chi_1(G) = 2n$. On the other hand, it follows from the corollary to Theorem 2 that

$$\chi_n(G) \geqslant \chi_1(G) + 2n - 2 = 2 + 2n - 2 = 2n.$$

$$\therefore \quad \chi_n(G) = 2n.$$

Q.E.D.

Both trees and cycles of even length are bipartite, so that Theorem 5 gives us all of their chromatic numbers. The derivation of the chromatic numbers of the odd cycles is not quite so easy and requires some preliminary discussions and lemmas.

It is well known that $\chi_1(C_{2p+1}) = 3$. Figures 2 and 3 show that $\chi_2(C_7) \leqslant 5$ and $\chi_3(C_7) \leqslant 7$. Theorem 2 enables us to change these into the equations $\chi_2(C_7) = 5$ and $\chi_3(C_7) = 7$. On the other hand, a 4-tuple coloring of C_7 with nine colors can not be found so that $\chi_4(C_7) \geqslant 10$. Applying Theorem 1 we get $\chi_4(C_7) \leqslant \chi_3(C_7) + \chi_1(C_7) = 7 + 3 = 10$. We are thus led to conjecture that $\chi_k(C_{2p+1}) = 2k + 1$ if $k \leqslant p$ and $\chi_{p+1}(C_{2p+1}) = 2p + 4$.

We know that an n-tuple coloring of C_{2p+1} with m colors is equivalent to a homomorphism $\theta\colon C_{2p+1} \to G_n{}^m$. As $\chi_1(C_{2p+1}) = 3$, we must also have $\chi_1(\theta(C_{2p+1})) \geqslant 3$. By a well-known theorem (see [3, p. 127]), $\theta(C_{2p+1})$ must contain an odd cycle whose length is obviously at most $2p + 1$. Thus, information regarding the shortest odd cycle of $G_n{}^m$ should be useful in determining whether a homomorphism of C_{2p+1} into $G_n{}^m$ can exist.

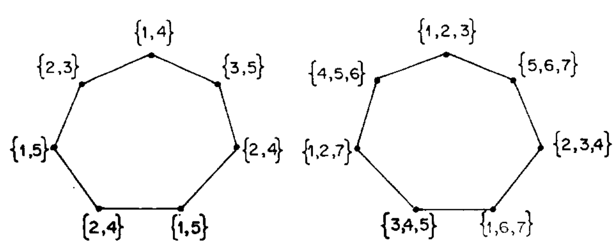
LEMMA 1. *Suppose $A, B \in V(G_n{}^m)$, and suppose further that they are joined by a path of length $2p$ ($p \geqslant 0$) in $G_n{}^m$, then*

$$| A \cap B | \geqslant n - (m - 2n)\, p.$$

Proof. (The author is indebted to the referee for this elegant proof.) We may assume that $p \geqslant 1$. Let $A = X_0, X_1, ..., X_{2p} = B$ be a path of length $2p$ in $G_n{}^m$. This implies that $X_i \cap X_{i+1} = \varnothing$, $0 \leqslant i \leqslant 2p - 1$. Now, for $0 \leqslant i \leqslant p - 1$ we have

$$\begin{aligned} | X_i \cup X_{2p-i} | &\leqslant |(I^m - X_{i+1}) \cup (I^m - X_{2p-i-1})| \\ &= m - | X_{i+1} \cap X_{2p-i-1} | \\ &= m - 2n + | X_{i+1} \cup X_{2p-i-1} |. \end{aligned}$$

$$\therefore \quad | X_0 \cup X_{2p} | \leqslant (m - 2n)(p - 1) + | X_{p-1} \cup X_{p+1} |.$$



FIGURES 2, 3

Thus, we conclude

$$\begin{aligned}
|A \cap B| = |X_0 \cap X_{2p}| &= |X_0| + |X_{2p}| - |X_0 \cup X_{2p}| \\
&\geqslant 2n - (m - 2n)(p - 1) - |X_{p-1} \cup X_{p+1}| \\
&\geqslant 2n - (m - 2n)(p - 1) - |I^m - X_p| \\
&= n - (m - 2n)\,p.
\end{aligned}$$

Q.E.D.

LEMMA 2. *If $G_n{}^m$ contains a $2p + 1$ cycle and if $m \geqslant 2n + 1$, then $2p + 1 \geqslant m/(m - 2n)$.*

Proof. Let $A_1, A_2, \ldots, A_{2p+1}$ be a $2p + 1$ cycle in $G_n{}^m$. Then $A_1, A_2, \ldots, A_{2p+1}$ is a $2p$ path joining A_1 and A_{2p+1}.

$$\therefore \quad |A_1 \cap A_{2p+1}| \geqslant n - (m - 2n)\,p.$$

On the other hand, A_1 and A_{p+1} are also adjacent in $G_n{}^m$, so

$$A_1 \cap A_{2p+1} = \varnothing.$$
$$\therefore \quad n - (m - 2n)\,p \leqslant 0,$$

and recalling that $m > 2n$, we get

$$p \geqslant n/(m - 2n) \qquad \text{or} \qquad 2p + 1 \geqslant m/(m - 2n).$$

Q.E.D.

It is to be noted that this lemma does not hold for even cycles. In $G_4{}^9$ the vertices $\{1, 2, 3, 4\}$, $\{6, 7, 8, 9\}$, $\{2, 3, 4, 5\}$, $\{1, 6, 7, 8\}$, $\{2, 3, 4, 9\}$, and $\{5, 6, 7, 8\}$ span a 6-cycle.

We next show that in the case $m = 2n + 1$, the lower bound on odd cycle lengths obtained in the previous lemma is actually attained.

LEMMA 3. *G_n^{2n+1} contains a $2n + 1$ cycle.*

Proof. For every positive integer i we define $A_i = \{(i - 1)\,n + j,\ 1 \leqslant j \leqslant n\}$, where addition is mod $2n + 1$. Clearly, if $i \equiv i' \bmod 2n + 1$ then $A_i = A_{i'}$. The reader can easily convince himself that the converse also holds, i.e., $A_i = A_{i'}$ iff $i \equiv i' \bmod 2n + 1$. Consequently, $\{A_i\}$ consists of $2n + 1$ vertices of G_n^{2n+1}. As $A_i \cap A_{i+1} = \varnothing$ they are adjacent in G_n^{2n+1} and so $\{A_i\}$ spans a subgraph of G_n^{2n+1} which contains a $2n + 1$ cycle. Figure 3 is an example of such a cycle. Q.E.D.

We are now ready to derive all the chromatic numbers of C_{2p+1}.

THEOREM 6.

$$\chi_n(C_{2p+1}) = 2n + 1 + [(n-1)/p].$$

Proof. It is well known that for any positive integer n, there exist unique integers k, r such that

$$n - 1 = kp + r \qquad k \geqslant 0 \qquad \text{and} \qquad p > r \geqslant 0.$$

The theorem will be proved by induction on k. We begin with $k = 0$. We already know that $\chi_1(C_{2p+1}) = 3$, and hence, by the corollary to Theorem 2 we have

$$\chi_p(C_{2p+1}) \geqslant 3 + 2p - 2 = 2p + 1.$$

On the other hand, since the inclusion of C_{2p+1} in G_p^{2p+1} (as given by Lemma 3 above) may be considered as a homomorphism of C_{2p+1} into G_p^{2p+1}, it follows that $\chi_p(C_{2p+1}) \leqslant 2p + 1$. Combining this with the above, we get

$$\chi_p(C_{2p+1}) = 2p + 1. \tag{8}$$

Applying the same corollary twice for each r $(1 < r < p)$, we get

$$\chi_p(C_{2p+1}) \geqslant \chi_r(C_{2p+1}) + 2p - 2r,$$

and

$$\chi_r(C_{2p+1}) \geqslant \chi_1(C_{2p+1}) + 2r - 2.$$

Hence, by (8),

$$\chi_r(C_{2p+1}) = 2r + 1 \qquad \text{for all } 1 \leqslant r \leqslant p.$$

This completes the proof for the case $k = 0$.

Assume now that the theorem has been proved for $k - 1$ and all $0 \leqslant r < p$. We will show that it also holds for k and all $0 \leqslant r < p$. Two cases are to be considered.

Case 1. $r = 0$. Here $[(n-1)/p] = k$ and $[(n-2)/p] = k - 1$. Now, if it were the case that $\chi_n(C_{2p+1}) < 2n + 1 + [(n-1)/p]$, we would have that $\chi_n(C_{2p+1}) \leqslant 2n + k = 2kp + k + 2$. Hence, by AF1, there would exist a homomorphism

$$\theta\colon C_{2p+1} \to G_{kp+1}^{2pk+k+2}.$$

As we have noted before, $\theta(C_{2p+1})$ must contain an odd cycle of length not greater than $2p + 1$. Lemma 2 now gives us

$$2p + 1 \geqslant (2kp + k + 2)/(2kp + k + 2 - 2kp - 2) = 2p + 1 + (2/k),$$

which is impossible. Hence, we must have

$$\chi_n(C_{2p+1}) \geqslant 2n + 1 + [(n-1)/p]. \tag{9}$$

On the other hand, it follows from Theorem 1 that $\chi_n(C_{2p+1}) \leqslant \chi_1(C_{2p+1}) + \chi_{n-1}(C_{2p+1})$. Since $n - 1 = kp$, we have $(n-1) - 1 = (k-1)p + (p-1)$, and we may apply the induction hypothesis to $\chi_{n-1}(C_{2p+1})$.

$$\begin{aligned} \therefore \quad \chi_n(C_{2p+1}) &\leqslant \chi_1(C_{2p+1}) + \chi_{n-1}(C_{2p+1}) \\ &= 3 + 2(n-1) + 1 + [(n-2)/p] \\ &= 2n + 2 + k - 1 = 2n + 1 + [(n-1)/p]. \end{aligned}$$

This, together with (9) above completes the proof for the case $r = 0$.

Case 2. $1 \leqslant r < p$. Theorem 1 gives us the upper bound

$$\chi_n(C_{2p+1}) \leqslant \chi_{n-p}(C_{2p+1}) + \chi_p(C_{2p+1}).$$

Now, $n - p - 1 = (k-1)p + r$, and hence, by the induction hypothesis,

$$\chi_{n-p}(C_{2p+1}) = 2(n-p) + 1 + [(n-p-1)/p] = 2(n-p) + k.$$

This and (8) now give us

$$\chi_n(C_{2p+1}) \leqslant 2(n-p) + k + 2p + 1 = 2n + 1 + [(n-1)/p].$$

We apply the corollary to Theorem 2 to obtain the lower bound.

$$\begin{aligned} \chi_n(C_{2p+1}) = \chi_{pk+1+r}(C_{2p+1}) &\geqslant \chi_{kp+1}(C_{2p+1}) \\ &\quad + 2(kp + 1 + r) - 2(kp + 1) \\ &= 2(kp+1) + 1 + [(kp + 1 - 1)/p] + 2n - 2(kp+1) \\ &= 2n + 1 + k = 2n + 1 + [(n-1)/p]. \end{aligned}$$

So again $\chi_n(C_{2p+1}) = 2n + 1 + [(n-1)/p]$, and the induction is complete. Q.E.D.

The following corollary characterizes the graphs for which equality holds in Theorem 2.

COROLLARY. *If there exists n_0 such that $\chi_{n+1}(G) = 2 + \chi_n(G)$ for all $n \geqslant n_0$, then G is bipartite and $\chi_{n+1}(G) = 2 + \chi_n(G)$ for all n.*

Proof. Clearly, if $n \geqslant n_0$, we have $\chi_n(G) = \chi_{n_0}(G) + 2n - 2n_0$. If

G is not bipartite then it must contain an odd cycle, say C_{2p+1}. So, $\chi_n(G) \geqslant \chi_n(C_{2p+1})$. Combining these two facts we now have

$$\chi_{n_0}(G) + 2n - 2n_0 \geqslant 2n + 1 + [(n-1)/p]$$

or

$$\chi_{n_0}(G) - 2n_0 \geqslant 1 + [(n-1)/p] \qquad \text{for all} \quad n \geqslant n_0 .$$

This, for sufficiently large n, is clearly impossible. Hence, G must be bipartite, and by Theorem 5, $\chi_{n+1}(G) = 2 + \chi_n(G)$ for all n. Q.E.D.

Theorem 6 can be used to calculate the chromatic numbers of a variety of graphs. We demonstrate this with an example and another theorem.

EXAMPLE. Figure 4 contains 1-tuple and 2-tuple colorings of a graph G, so $\chi_1(G) \leqslant 3$ and $\chi_2(G) \leqslant 5$. Since G contains a 5-cycle we must have $\chi_n(G) \geqslant \chi_n(C_5) = 2n + 1 + [(n-1)/2]$. On the other hand,

$$\chi_{2m}(G) \leqslant m\chi_2(G) \leqslant 5m = 2(2m) + 1 + [(2m-1)/2],$$

and

$$\begin{aligned}\chi_{2m+1}(G) &\leqslant m\chi_2(G) + \chi_1(G) \leqslant 5m + 3 \\ &= 2(2m+1) + 1 + [(2m+1-1)/2].\end{aligned}$$

And combining the above inequalities we have

$$\chi_n(G) \leqslant 2n + 1 + [(n-1)/2].$$

Since the reverse inequality was derived before, we have

$$\chi_n(G) = 2n + 1 + [(n-1)/2].$$

The same method can be applied to derive the chromatic numbers of G_m^{2m+1}.

THEOREM 7. $\chi_n(G_m^{2m+1}) = 2n + 1 + [(n-1)/m]$.

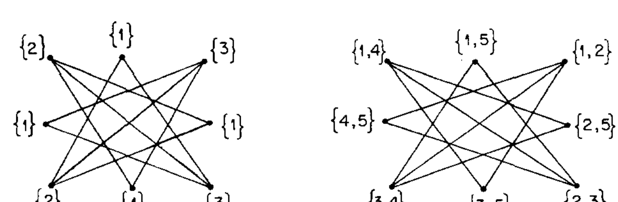


FIGURE 4

Proof. We already know from Lemma 3 of Theorem 6 that G_m^{2m+1} contains a $2m+1$ cycle. Consequently we have the lower bound

$$\chi_n(G_m^{2m+1}) \geqslant \chi_n(C_{2m+1}) = 2n + 1 + [(n-1)/m]. \tag{10}$$

The reverse inequality will again be proved by induction on k where $n - 1 = km + r$, $k \geqslant 0$, $m > r \geqslant 0$. Suppose $k = 0$. By Theorem 2, $\chi_n(G_m^{2m+1}) \leqslant \chi_m(G_m^{2m+1}) - 2(m-n)$, $1 \leqslant n \leqslant m$. Since the identity map $G_m^{2m+1} \to G_m^{2m+1}$ is a homomorphism, $\chi_m(G_m^{2m+1}) \leqslant 2m + 1$. This completes the proof for the case $k = 0$ and $1 \leqslant n \leqslant m$. Assume now that the theorem holds for $k - 1$. As $n - m - 1 = (k-1)m + r$, we may apply the induction hypothesis to $n - m$, so that we have

$$\begin{aligned}\chi_n(G_m^{2m+1}) &\leqslant \chi_{n-m}(G_m^{2m+1}) + \chi_m(G_m^{2m+1}) \\ &= 2(n-m) + 1 + [(n-m-1)/m] + 2m + 1 \\ &= 2n + 1 + [(n-1)/m]. \end{aligned}$$

Q.E.D.

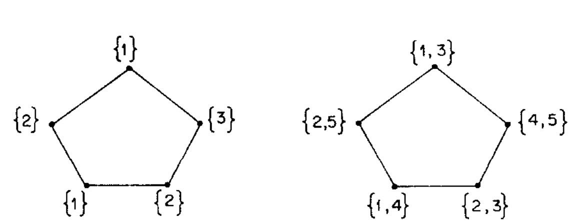
4. Efficient Colorings

Figures 5 and 6 below show examples of minimal 1-tuple and 2-tuple colorings of C_5. The color classes in Fig. 5 consist of two maximal independent subsets and a singleton. The color classes in Fig. 6 consist of five maximal independent subsets. Thus we might say that the 2-tuple coloring makes "better" use of the independent subsets of C_5. This notion is made precise by the following definition.

Definition. An n-tuple coloring of G with m colors is said to be *efficient* if

$$m/n \leqslant (\chi_{k(G)})/k \qquad \text{for all } k \geqslant 1.$$

It is clear that any efficient coloring is necessarily minimal. On the other hand, it is not known whether every graph has an efficient coloring. Theorem 8 gives a sufficient condition for the existence of such efficient



Figures 5, 6

colorings. We first recall that an independent set is said to be *maximum* if its cardinality is not less than that of any other independent subset of the same graph.

THEOREM 8. *If $\{C_\lambda\}$ is an n-tuple coloring of G with m colors, and if each C_λ is a maximum independent subset of $V(G)$, then $\{C_\lambda\}$ is an efficient coloring of G. Moreover, all efficient colorings of G must then consist of maximum independent sets.*

Proof. Let $\beta_0 = | C_\lambda |$ for all λ, $p = | V(G)|$ and suppose D_ν is a minimal k-tuple coloring of G. It follows from AF2 that

$$\sum_{\lambda=1}^{m} | C_\lambda | = pn \qquad \text{and} \qquad \sum_{\nu=1}^{\chi_k(G)} | D_\nu | = pk.$$

Since $\beta_0 = | C_\lambda | \geqslant | D_\nu |$ for all λ, ν, we have

$$\beta_0 m = pn \qquad \text{and} \qquad \beta_0 \chi_k(G) \geqslant pk \tag{11}$$

or

$$m/n = p/\beta_0 \leqslant (\chi_{k(G)})/k.$$

Hence $\{C_\lambda\}$ is an efficient coloring. Moreover, $m/n = (\chi_{k(G)})/k$ iff equality holds in (11); in other words iff each D_ν is maximum. Q.E.D.

Before we proceed to derive the efficient colorings of the $G_n{}^m$, we must digress and examine the independent subsets of $V(G_n{}^m)$ closely. We know that two vertices of $G_n{}^m$ are not adjacent iff they are not disjoint. Hence, an independent set of vertices of $G_n{}^m$ consists of a family $\{a_\lambda\}$ of pairwise nondisjoint subsets of I^m such that $| a_\lambda | = n$ for all λ. Such families have already been investigated by other authors (see [1, 4]) and are said to have the *intersection property*. We follow the notation of [4] where $S(1, n, m)$ denotes, in effect, the family of all independent sets of vertices of $G_n{}^m$, and write,

THEOREM OF ERDÖS–CHAO KO–RADO. *If* $1 \leqslant n \leqslant \frac{1}{2}m$ *and*

$$\{a_\lambda\} \in S(1, n, m), \qquad \textit{then} \qquad |\{a_\lambda\}| \leqslant \binom{m-1}{n-1}.$$

Now [4, Remark 2, p. 334] is not quite valid, as equality can hold even if $m > 2l - k$. This can happen if $\{a_\lambda\}$ does not happen to contain all the available integers, i.e., if $\bigcup_\lambda a_\lambda \neq \{0, 1, 2,..., m-1\}$. The crucial point in the proof of [4, Theorem 2] occurs on p. 332, the sentence "If $n_0 = 0$,..." For by applying the induction hypothesis we may carry over equality to the case $m > 2l - k$. This point, minor as it may be, enables us to strengthen the above theorem.

THEOREM OF ERDÖS–CHAO KO–RADO (Strong Version). *If* $1 \leqslant n \leqslant \frac{1}{2}m$ *and* $\{a_\lambda\} \in S(1, n, m)$, *then* $|\{a_\lambda\}| \leqslant \binom{m-1}{n-1}$ *and if equality holds and* $m > 2n$, *then* $|\bigcap_\lambda a_\lambda| = 1$.

Proof. We will merely indicate the required additions to Katona's proof of [4, Remark 3, p. 334]. If $|\{a_\lambda\}| = \binom{m-1}{n-1}$, then equality must hold in (10). Hence, by the corrected version of Remark 2 given above, either $m = 2(m - l) - (m - 2l + 1)$, or $m - n = n$, or $\bigcup_\nu b_\nu \neq \{1, 2,..., m\}$. The first alternative is manifestly self-contradictory and the second contradicts $m > 2n$. Consequently, the third alternative must hold.

$$\therefore \quad \bigcap_\lambda a_\lambda \neq \varnothing .$$

However, if $|\bigcap_\lambda a_\lambda| = k$, then $|\{a_\lambda\}| \leqslant \binom{m-k}{n-k}$, and hence $k = 1$.

Q.E.D.

It now follows easily from the strenghtened version of the Erdös–Chao Ko–Rado theorem that the maximum independent sets of vertices of $G_n{}^m$ are the sets C_i where $C_i = \{A \in V(G_n{}^m) \mid i \in A\}$. The derivation of the efficient colorings of $G_n{}^m$ is now quite an easy matter.

THEOREM 9. *If* $m > 2n$, *then* $G_n{}^m$ *has an efficient k-tuple coloring iff n divides k.*

Proof. Let C_i be as defined above. Clearly $\{C_i\}$ for $1 \leqslant i \leqslant m$ is an n-tuple coloring of $G_n{}^m$. By Theorem 8 this coloring is efficient. Moreover, writing χ_i for $\chi_i(G_n{}^m)$,

$$\begin{gathered} \chi_n/n = m/n \leqslant \chi_{sn}/sn \leqslant s\chi_n/sn = \chi_n/n \qquad \text{for all} \quad s \geqslant 1. \\ \therefore \quad \chi_{sn}/sn = \chi_n/n = m/n. \end{gathered} \tag{12}$$

In other words, if n divides k, then $G_n{}^m$ has an efficient k-tuple coloring. Conversely, suppose $G_n{}^m$ has a k-tuple efficient coloring $\{D_\lambda\}$. By Theorem 8, each D_λ is a maximum independent set. By the strong Erdös–Chao Ko–Rado theorem, for each D_λ there is an integer j_λ such that $j_\lambda \in A$ for all $A \in D_\lambda$. Let ν_j be the number of D_λ for which $j_\lambda = j$. Now, a vertex A of $G_n{}^m$ is contained in D_λ iff $j_\lambda \in A$. Since each vertex is contained in k of the D_λ, it follows that

$$\sum_{j \in A} \nu_j = k.$$

For any two integers h and i, where $1 \leqslant h \neq i \leqslant n$, choose vertices A and B so that they differ only by h and i; in other words, choose A and B so that $h \in A$, $i \in B$ and $A = B - \{i\} + \{h\}$. Then

$$0 = k - k = \left(\sum_{j \in A} \nu_j\right) - \left(\sum_{s \in B} \nu_s\right) = \nu_h - \nu_i \,.$$

So,

$$\nu_h = \nu_i \qquad \text{for all } 1 \leqslant h, i \leqslant n.$$

Hence, if ν is the common value of all the ν_j, then each vertex of $G_n{}^m$ is contained in $n\nu$ of the D_λ.

$$\therefore \quad k = n\nu \qquad \text{and} \qquad n \text{ divides } k.$$

Q.E.D.

COROLLARY. $\chi_{kn}(G_n{}^m) = km$.

Proof. See (12). Q.E.D.

Efficient colorings can be used to derive the chromatic numbers of nonsymmetric graphs as well. Consider, for example, the graph G of Fig. 7. Clearly $\chi_1(G) = 4$. The reader can easily convince himself, applying Theorem 8, that the 2-tuple coloring given in Fig. 7 is in fact efficient. Hence, by Theorem 9, $\chi_{2n}(G) = 7n$. Now, by Theorem 1, $\chi_{2n+1}(G) \leqslant \chi_1(G) + \chi_{2n}(G) = 4 + 7n$. Since the 2-tuple coloring is efficient we must also have

$$(\chi_{2n+1}(G))/(2n + 1) \geqslant 7/2.$$

$$\therefore \quad \chi_{2n+1}(G) \geqslant 7n + (7/2).$$

And since $\chi_{2n+1}(G)$ is an integer we must have $\chi_{2n+1}(G) = 7n + 4$.

Theorems 8 and 9 can be used to calculate some more of the numbers $\chi_k(G_n{}^m)$. However, before proceeding further with these calculations, the reader should be made aware of the significance of these numbers. This

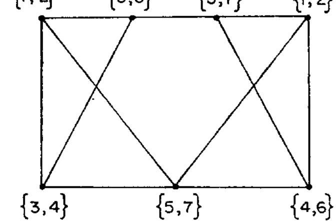


FIGURE 7

can best be done with the aid of a concrete example. It follows from Theorem 7 that $\chi_6(G_4{}^9) = 14$. Hence, there exists a homomorphism $G_4{}^9 \to G_9^{14}$. In view of AF1 it now follows that if any graph can be given a 4-tuple coloring with nine colors, then at most 14 colors are required for a 6-tuple coloring. Thus the numbers $\chi_k(G_n{}^m)$ contain valuable information relating different chromatic numbers. The proof of Theorem 10 below serves as an example for the utilization of this information. First, however, a lemma is needed.

LEMMA. *If $\chi_k(G_n{}^m) = \chi_k(G_t{}^s)$ for all k, then $m = s$ and $n = t$.*

Proof. Since $G_n{}^m$ has an n-tuple efficient coloring and $G_t{}^s$ has a t-tuple efficient coloring, it follows from Theorem 9 that n divides t, and t divides n.

$$\therefore \quad n = t.$$

Moreover,

$$m = \chi_n(G_n{}^m) = \chi_n(G_t{}^s) = \chi_t(G_t{}^s) = s.$$

Q.E.D.

THEOREM 10. $\chi_k(G_n^{2n+2}) = 2 + 2k$ *if* $(n-1)/2 < k \leqslant n$.

Proof. If $k = n$, then this theorem reduces to a special case of the corollary to Theorem 9. Hence, we may assume that $(n-1)/2 < k < n$, or,

$$[(n-1)/k] = 1.$$

By Theorem 7, $\chi_n(G_k^{2k+1}) = 2n + 1 + [(n-1)/k] = 2n + 2$, and so there exists a homomorphism

$$G_k^{2k+1} \to G_n^{2n+2}. \tag{13}$$

Suppose now that $\chi_k(G_n^{2n+2}) \leqslant 2k + 1$, then there exists a homomorphism

$$G_n^{2n+2} \to G_k^{2k+1}. \tag{14}$$

It now follows from Theorem 3 and the lemma (applied to (13) and (14)) that $2n + 2 = 2k + 1$ and $n = k$, which is absurd. Hence, we must have $\chi_k(G_n^{2n+2}) \geqslant 2k + 2$. The upper bound is given to us by the corollary to Theorem 2:

$$2n + 2 = \chi_n(G_n^{2n+2}) \geqslant \chi_k(G_n^{2n+2}) + 2n - 2k,$$

or

$$\chi_k(G_n^{2n+2}) \leqslant 2k + 2. \qquad \text{Q.E.D.}$$

A function $f: I^+ \to I^+$ is said to be *subadditive* if $f(a+b) \leqslant f(a) + f(b)$. Theorem 1 states that $\chi_n(G)$ is a subadditive function of its argument n. The following theorem on subadditive functions again demonstrates the significance of efficient colorings.

THEOREM 11. *If $f: I^+ \to I^+$ is a subadditive function, and if there exists n such that*

$$f(x)/x \geqslant f(n)/n \qquad \textit{for all } x \geqslant 1,$$

then there exists an integer x_0 such that

$$f(x+n) = f(x) + f(n) \qquad \textit{for all } x \geqslant x_0 .$$

Proof. Since f is subadditive we know that $f(kn) \leqslant kf(n)$. But we are also given that

$$f(kn)/kn \geqslant f(n)/n.$$
$$\therefore \quad f(kn) = kf(n).$$

Now, for arbitrary k and $0 \leqslant r < n$, define

$$\Delta(k, r) = f(kn + r) - f(kn).$$

Then

$$\begin{aligned} \Delta(k, r) - \Delta(k+1, r) &= f(kn+r) - f(kn) - f((k+1)\,n + r) \\ &\quad + f((k+1)\,n) \\ &= f(n) + f(kn+r) - f(kn+r+n) \geqslant 0. \end{aligned} \tag{15}$$

The last part of the above inequality is due to the subadditivity of f. On the other hand,

$$\begin{aligned} f(kn+r) + f(n-r) \geqslant f((k+1)\,n) &= (k+1)\,f(n) = kf(n) + f(n) \\ &= f(kn) + f(n) \end{aligned}$$
$$\therefore \quad f(kn+r) - f(kn) \geqslant f(n) - f(n-r),$$

or,

$$\Delta(k, r) \geqslant f(n) - f(n-r) \qquad \text{for all } k.$$

Hence,

$$\Delta(k, r) \geqslant \Delta(k+1, r) \geqslant f(n) - f(n-r) \qquad \text{for all } k,$$

and as $\Delta(k, r)$ can assume only integral values, it follows that there exists an integer m_r such that $\Delta(k, r) = \Delta(k+1, r)$ whenever $k \geqslant m_r$. Let $M = \operatorname{Max}\{m_r\}$, then,

$$\text{for} \quad k \geqslant M \quad \Delta(k, r) = \Delta(k+1, r) \quad \text{for all } 0 \leqslant r < n.$$

Set $x_0 = Mn$. If $x \geqslant x_0$ then there exist integers k and r, satisfying $k \geqslant M$ and $0 \leqslant r < n$, such that $x = kn + r$. Consequently, $\Delta(k, r) = \Delta(k+1, r)$, and equality must hold in (15).

$$\therefore \quad f(n) + f(kn + r) - f(kn + r + n) = 0.$$

In other words,

$$f(x) + f(n) = f(x + n).$$

Q.E.D.

Corollary. *For a fixed G_n^m and sufficiently large k,*

$$\chi_{k+n}(G_n^m) = m + \chi_k(G_n^m).$$

Proof. Merely apply Theorems 9 and 11 to G_n^m. Q.E.D.

5. Conclusion

A pattern for $\chi_k(G_n^m)$ emerges from the theorems of this paper and several other calculations not displayed here. It would seem that in general $\chi_1(G_n^m) = m - 2n + 2$, and consequently, by Theorem 2,

$$\chi_k(G_n^m) = 2 + \chi_{k-1}(G_n^m) \quad \text{for} \quad 1 < k \leqslant n.$$

For $k = n + 1$ it seems again that $\chi_{n+1}(G_n^m) = \chi_n(G_n^m) + \chi_1(G_n^m)$. Again, since $\chi_{2n}(G_n^m) = 2m$, this would entail

$$\chi_k(G_n^m) = 2 + \chi_{k-1}(G_n^m) \quad \text{for} \quad n + 1 < k \leqslant 2n.$$

We generalize this to the following.

Conjecture. If $k - 1 = qn + r$ where $q \geqslant 0$ and $0 \leqslant r < n$, then

$$\chi_k(G_n^m) = (q + 1)\, m - 2(n - r - 1).$$

Added in Proof. The author wishes to thank the referee for the current version of Theorem 11, as well as for many other helpful suggestions. In addition, it should be pointed out that it has recently been proved in [5] and [6] that every graph does indeed possess efficient colorings. In fact, a slightly different version of Theorem 11 is known to Scott [5], whose as yet unpublished results predate those presented here.

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Printed in Belgium